a-T-menability of groups acting on trees

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Abstract. We present some partial results concerning a-T-menability of groups acting on trees. Various known results are given uniform proofs.

1. Definition of a-T-menability.

Definition 1.1. *Given a metric space (X,ρ), the action* $\mathsf{G}\to\mathrm{Isom}(X)$ *is* metrically proper *if given $x\in X$ the displacement function $\mathsf{G}\ni g\mapsto\rho(x,gx)\in\mathbb{R}$ is proper. If the group action is set we say that X is metrically proper.*

The above property does not depend on a choice of a point x.

Definition (M. Gromov) 1.2. *A locally compact, second countable, compactly generated group* G *is* a-T-menable *if there exists metrically proper isometric* G*-action on some affine Hilbert space.*

Remark : a-T-menability is equivalent to the existence of C_0-approximate unity consisting of positive definite functions. The latter is called in the literature the Approximation Property of Haagerup.

Throughout this paper by representation we mean isometric affine action. By a subgroup we always mean a closed subgroup.

2. Motivation.

We are motivated by the following Theorems:

Theorem 2.1. *[JJV, Th. 6.2.8] Let Γ be a countable group acting on a tree without inversions, with finite edge stabilizers. If vertex stabilizers in Γ are a-T-menable, then so is Γ.*

Theorem 2.2. *[JJV, Ex. 6.1.6] If* N *is a-T-menable and* G/N *is amenable then* G *is a-T-menable.*

On the other hand:

Theorem 2.3. *[HV, Ch. 8 L. 6] Every isometric affine representation of $G=SL_2\mathbb{Z}\ltimes\mathbb{Z}^2$ has a $\mathbb{Z}^2$-fixed point.*

The group G acts on a tree as it can be decomposed as $(\mathbb{Z}/4\ltimes\mathbb{Z}^2)*_{\mathbb{Z}/2\ltimes\mathbb{Z}^2}(\mathbb{Z}/6\ltimes\mathbb{Z}^2)$. The factors are a-T-menable by Theorem 2.2 , since $\mathbb{Z}^2$ is a-T-menable as it acts on $\mathbb{R}^2$. On the other hand $Sl_2\mathbb{Z}$ is a-T-menable by the result of Haagerup [H]. Therefore extra assumptions about finitness of edge stabilizers (in Theorem 2.1) or amenability of the quotients (in Theorem 2.2) cannot be simply weakened to a-T-menability.

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We believe that a-T-menability is a property of representations (see next section for definitions), rather than that of groups. Therefore in this mood we will concentrate on a question: what are the conditions under which given proper affine representations of two groups extend to one of their product with amalgamation, rather than whether there exist some such representation that extend.

The most naïve observation is that if there is a proper representation of a free product with amalgamation, the restrictions are proper representations of the factors that coincide on the common subgroup. It is not known whether the reverse holds, however we will prove some results in this direction. We would like to thank Tadeusz Januszkiewicz for calling our attention to it.

The main result of this paper is Theorem 5.4. As a result we strengthen a result from [JJV] (see Section 6). Constructions, we give, are purely geometric. We will also give an affirmative answer to the question of A. Valette, whether Baumslag-Solitar groups are a-T-menable. This was originally done (using another approach) in [GJ].

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3. **Actions on trees in general.**

Before we examine the case of a free product with amalgamation, let us restate a general observation of Haagerup [H] and its easy generalizations.

Let T be a tree. By E we denote a vector space of functions on edges of T with finite support. By V we denote the affine space of functions on vertices of T with finite support and total mass one. The structure of affine space is given as follows: $\delta_v - \delta_w$ is equal to the characteristic function of the segment joining vertices v and w (with the appropriate signs with respect so some auxiliary orientation on the edges of T).

Definition 3.1. *Let $U(T)$ be an affine Hilbert space completion of V defined above.*

There is an obvious $Aut(T)$ action of the group of cellular automorphisms of T on $U(T)$, with an $Aut(T)$-equivariant isometric embedding of T. An immediate consequence of the construction is

Proposition 3.2. *If Γ acts metrically properly on a tree T then Γ is a-T-menable.*

In particular $Sl_2\mathbb{Z} = \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ is a-T-menable.

If T is locally finite, the (topological) group of all cellular automorphisms of T is a-T-menable. Even if Γ acts effectively on T, the inclusion $\Gamma \to$Aut(T) in general is not closed (therefore we cannot conclude that Γ is a-T-menable). However we have

Proposition 3.3. *If Γ acts on a locally finite tree T and there exists an affine representation of Γ on an affine Hilbert space W, such that stabilizer of any vertex acts properly, then $U(T) \oplus W$ is Γ-proper.*

Proof : Given $g_n \in \Gamma$ with bounded displacement (as acting on T), for any vertex v the distance from $g_n v$ to v is bounded. Since there are only finitely many such vertices, one can take a subsequence such that $g_n v$ is constant. Stabilizers of vertices act properly on W, thus $\{g_n\}$ is relatively compact. $\square$

Proposition 3.4. *If Γ acts on a locally finite tree T and for each vertex v some affine proper representation of $Stab_v$ on W_v extends to a (perhaps non-proper) affine representation of Γ, then Γ is a-T-menable.*

Proof : This is obvious if the quotient $\Gamma\backslash T$ has finitely many vertices, since then the sum of the representations corresponding to any lifts will fulfill the assumptions of Proposition 3.3

Since there is no infinite sum operation in the category of affine Hilbert spaces, the construction will depend on choices made.

Let $\{v_k\}$ be a sequence of representatives of the vertices of the quotients. Let K_k be an exhausting sequence of compact subsets of Γ, let $x_k \in W_{v_k}$, $a_k = k^2+\sup\{||x_k - gx_k||^2 \colon g \in K_k\}$. Define a norm on $\prod_k W_{v_k}$ in a following way: $||y-z||^2 \colon =\colon \sum a_k^{-2}||y_k - z_k||^2$. Let W be the completition of the afiine space $\{y \in \prod_k W_{v_k} \colon ||y - x|| < \infty\}$

The choices are made in a such way that Γ acts diagonally on W and there are Γ-equivariant projections (up to scalar change of norm) to each of W_{v_k}. □

Note : The proof of Proposition 3.4 follows the standard proof of the fact that direct limit of a-T-menable groups is also a-T-menable.

The example of $Sl_2\mathbb{Z}\ltimes\mathbb{Z}^2$ shows that there are some obstructions for a representation to extend from the subgroup. In the terms of group cohomology, inclusion of groups need not induce epimorphisms on the level of the first cohomology.

4. Affine representations and subgroups.

If V is G-invariant subspace of W then W/V is a linear G-representation (the coset V is a fixed point), therefore V is metrically proper iff W is. In general, there is no minimal invariant subspace.

Let $\mathsf{H} < \mathsf{G}$. Assume $V \subset W$ is H-invariant subspace. Fix $x \in W$. Define $\psi_{(V\subset W)} \colon \mathsf{H}\backslash G \ni \mathsf{H}g \mapsto ||gx + V|| \in \mathbb{R}$ (where $gx + V$ is a coset of gx in W/V). If $\psi_{(V\subset W)}$ is proper we say that $(V \subset W)$ is $\mathsf{H}\backslash\mathsf{G}$-proper. The definition does not depend on the choice of x. If $\mathsf{H} = \{e\} < \mathsf{G}$, and V is any point, then $(V \subset W)$ is $\mathsf{H}\backslash\mathsf{G}$-proper exactly if W is G-proper.

Lemma 4.1. *Let $\mathsf{G}_1 < \mathsf{G}_2 < \mathsf{G}_3$. If there are $W_1 \subset W_2 \subset W_3$, such that W_i is G_i invariant and $(W_i \subset W_{i+1})$ is $\mathsf{G}_i\backslash\mathsf{G}_{i+1}$-proper then $(W_1 \subset W_3)$ is $\mathsf{G}_1\backslash\mathsf{G}_3$-proper.*

Proof : If $\psi_{(W_1\subset W_3)}(\mathsf{G}_1 g_n)$ is bounded, then $\psi_{(W_2\subset W_3)}(\mathsf{G}_2 g_n)$ is bounded. Therefore $g_n = g'_n h_n$ where $g'_n \in \mathsf{G}_2$ and $\{h_n\}$ is relatively compact. By the triangle inequality $||g'_n x - x|| \leq ||g_n(h_n^{-1}x - x)|| + ||g_n x - x|| + ||h_n^{-1}x - x||$, so $\psi_{(W_1\subset W_2)}(\mathsf{G}_1 g'_n)$ is bounded, therefore $g'_n = g''_n h'_n$, where $g''_n \in \mathsf{G}_1$ and $\{h'_n\}$ is relatively compact. Finally $g_n = g''_n(h'_n h_n)$. □

Unfortunately it is not known whether for any $\mathsf{H} < \mathsf{G}$ and proper G-representation W there exist H-invariant subspace V, such that $(V \subset W)$ is $\mathsf{H}\backslash\mathsf{G}$-proper. The cases when it does happend are discussed in the following sections.

5. Free products with amalgamation.

If $\Gamma = \mathsf{G}_1 *_{\mathsf{H}} \mathsf{G}_2$ then, according to Serre theory [S], a graph T with the set of vertices equal to $\Gamma/\mathsf{G}_1 \cup \Gamma/\mathsf{G}_2$ and the set of edges equal to Γ/H, with inclusion as incidence relation,

is a tree (on which Γ acts on the left). The representations of G_1 and G_2 on $U(T)$ have global fixed points.

Let H be a common subgroup in G_1 and G_2. Let W_i be G_i-representations. Let W be their common H-invariant subspace. Define $\mathcal{H}_i := W_i/W$. Inductively decompose $\uparrow_{\mathsf{H}}^{\mathsf{G}_i} \mathcal{H}_\omega$ (where ω is a sequence of 1s and 2s) with respect to H as $\mathcal{H}_\omega \oplus \mathcal{H}_{i\omega}$. G_1 acts on $\mathcal{H}_2^\bullet = (\mathcal{H}_2 \oplus \mathcal{H}_{12}) \oplus (\mathcal{H}_{212} \oplus \mathcal{H}_{1212}) \oplus \dots$ and $\mathcal{H}_1^\circ = (\mathcal{H}_{21} \oplus \mathcal{H}_{121}) \oplus \dots$, G_2 acts on $\mathcal{H}_1^\bullet = (\mathcal{H}_1 \oplus \mathcal{H}_{21}) \oplus \cdots$ and $\mathcal{H}_2^\circ = (\mathcal{H}_{12} \oplus \mathcal{H}_{212}) \oplus \cdots$. Both representations of H on $\mathcal{H}_i^\bullet$ coincide.

Definition 5.1. *Let* $W_\Gamma = W \oplus \mathcal{H}_1^\bullet \oplus \mathcal{H}_2^\bullet = W_1 \oplus \mathcal{H}_1^\circ \oplus \mathcal{H}_2^\bullet = W_2 \oplus \mathcal{H}_1^\bullet \oplus \mathcal{H}_2^\circ$.

An immediate consequence from the construction is

Theorem 5.2. *Let* $\mathsf{H} < \mathsf{G}_i$, $i = 1, 2$. *Let* W_i *be* G_i*-representations. Let* W *be their common* H*-invariant subspace. Let* $\Gamma = \mathsf{G}_1 *_{\mathsf{H}} \mathsf{G}_2$. *Then* W_i *and* W *are respectively* G_i- *and* $\mathsf{H}-$*invariant subspaces of* W_Γ.

Note : Although there is no way to induce an affine representation, W_Γ is morally equal to $\uparrow_{\mathsf{G}_1}^{\Gamma} W_1 \oplus \uparrow_{\mathsf{G}_2}^{\Gamma} W_2 / \uparrow_{\mathsf{H}}^{\Gamma} W$. If $W' \subset W$ is another H-invariant subspace, then W_Γ is not a Γ-invariant subspace of W'_Γ (constructed from the triple $W_1 \supset W' \subset W_2$).

A straightforward consequence of Proposition 3.3 and Theorem 5.2 is the following

Corollary 5.3. *If* H *is of finite index in* G_1 *and* G_2 *and if there are metrically proper representations of* G_i *that coincide when restricted to* H, *then* $\mathsf{G}_1 *_{\mathsf{H}} \mathsf{G}_2$ *is a-T-menable.*

Example [JJV,BCS]. The torus group $\Gamma_{p,q} = \langle x, y | x^p = y^q \rangle$ is a-T-menable.

Theorem 5.4. *With the assumptions of Theorem 5.2 , if* W_i *is proper affine* G_i*-representation and* $(W \subset W_i)$ *is* $\mathsf{H}\backslash\mathsf{G}_i$*-proper for* $i = 1, 2$ *then*
(1) $W_\Gamma \oplus U(T)$ *is* Γ*-proper,*
(2) if $\mathsf{H}' < \mathsf{G}_1$ *and* V *is proper* H'*-space such that* $(V \subset W_1)$ *is* $\mathsf{H}'\backslash\mathsf{G}_1$*-proper, then* $(V \subset W_\Gamma \oplus U(T))$ *is* $\mathsf{H}'\backslash\Gamma$*-proper.*

Proof : In fact (1) is a special case of (2). Therefore we will prove (2).

Given $\gamma_n \in \Gamma$ such that $\psi_{(W,V)}(\gamma_n)$ is bounded. Define *the length function* $\ell: \Gamma \to \mathbb{N}$ by $\ell_{|\mathsf{H}} \equiv 0$, $\ell(\gamma) = min\{\ell(\eta) + 1 | \gamma = \eta g$, where $g \in \mathsf{G}_1 \cup \mathsf{G}_2\}$. This function is equal to the distortion of the action on $U(T)$. Therefore we may find a subsequence such that $\ell(\gamma_n) = k$.

If $k \leq 1$ there is nothing to prove. If all but finitely many $\gamma_n \in \mathsf{G}_2$ we have to use Lemma 4.1

Let $x \in V$. Define $\varphi_x(\gamma)$ to be the component of γx in $\bigoplus_{|\omega|=l(\gamma)} \mathcal{H}_\omega$.

Lemma 5.5. *If* $\gamma = \gamma_1\gamma_2$, $\ell(\gamma) = \ell(\gamma_1) + \ell(\gamma_2)$ *and* $\ell(\gamma_2) \geq 1$, *then* $||\varphi_x(\gamma)|| = ||\varphi_x(\gamma_2)||$.

Proof : Without loss of generality $\ell(\gamma_1) = 1$. From the definition of induced representation $\gamma_1\mathcal{H}_\omega \perp \mathcal{H}_\omega$, therefore $\varphi_x(\gamma) = \gamma_1\varphi_x(\gamma_2)$. □

Now we proceed by induction on k as follows: we define η_n such that $\gamma_n = \eta_n g_n$ ($g_n \in \mathsf{G}_i$) and $l(\eta_n) = k - 1$.

From Lemma 5.5 we see $||\gamma_n x - x|| \geq ||\varphi_x(\gamma_n)|| = ||\varphi_x(g_n)|| = \psi_{(W_i,W)}(g_n)$. Therefore $\{g_n\}$ is relatively compact. Since $\eta_n x = \gamma_n x - \eta_n(g_n x - x)$ and, by induction assumption,

$\psi_{(V,W_\Gamma)}$ is proper when restricted to cosets of elements of length smaller than k, we obtain the claim. $\square$

6. Groups that act on trees with finite edge stabilizers.

The first case, where it is easy to fulfill the assumptions of the Theorem 5.4 is when H is finite, since then one can find a fixed point of any H-representation simply taking the center of mass $*$ of any orbit. The pair $(\{*\} \subset W_i)$ is $\mathsf{H}\backslash\mathsf{G}_i$ proper iff W_i is proper G_i-space. Summarizing this:

Proposition. *[JJV, Prop. 6.2.3 (1)] Let G_1, G_2 be two grups containing finite subgroup H, and let $\Gamma = \mathsf{G}_1 *_{\mathsf{H}} \mathsf{G}_2$ be the corresponging amalgamated product. If G_1 and G_2 are a-T-menable, then so is Γ.*

Theorem 2.1 is an easy consequence of the above [JJV].

7. Baumslag-Solitar groups ant their certain generalizations

The second easy case occurs when H is of finite index in G_i's. Then any pair is automatically $\mathsf{H}\backslash\mathsf{G}_i$ proper.

Let us recall some definitions from [GJ]. Let $\mathsf{G} \subset \mathfrak{N}$ be a closed subgroup of a locally compact compactly generated topological group $\mathfrak{N}$. Let $i_k\colon \mathsf{H} \to \mathsf{G}$ $(k = 1, 2)$ be two inclusions onto finite index open subgroups, which are conjugated by an automorphism ϕ of $\mathfrak{N}$.

Definition 7.1. *The $\mathfrak{N}$-BS group Γ is the group derived from $(\mathsf{G}, \mathsf{H}, i_1, i_2)$ by the (topological) HNN construction.*

Theorem 7.2. *[GJ] If $\mathfrak{N}$ is a-T-menable then $\mathfrak{N}$-BS groups are a-T-menable.*

Proof : We mimic the proof [JJV 6.2.7] for the case of a HNN extension, where the edge stabilizer is finite.

Step 1. Let Γ_0 be a fundamental group of the following tree:

$$\begin{tikzcd} \cdots & \mathsf{G} & & \mathsf{G} & & \mathsf{G} & \cdots \\ & & \mathsf{H} \arrow[ul, "i_1"] \arrow[ur, "i_2"'] & & \mathsf{H} \arrow[ul, "i_1"] \arrow[ur, "i_2"'] & & \end{tikzcd}$$

We have to find consistent representations of different copies of G. The Hilbert space in each case will be the one on which $\mathfrak{N}$ acts properly. The k-th copy acts by $\phi^k(\cdot)$, where ϕ is the automorphism of $\mathfrak{N}$ that conjugates i_1 and i_2.

By induction, each of

$$\begin{tikzcd} \mathsf{G} *_{\mathsf{H}} \cdots *_{\mathsf{H}} \mathsf{G} & & \mathsf{G} \\ & \mathsf{H} \arrow[ul, "i_1"] \arrow[ur, "i_2"'] & \end{tikzcd}$$

satisfies assumptions of Theorem 5.4 (alternatively: by Theorem 5.2 and induction we construct representation of Γ_0 and then use Proposition 3.4). It is easy to show [JJV

Prop. 6.1.1] that an increasing union of open a-T-menable subgroups is again a-T-menable. Therefore Γ_0 is a-T-menable.

Step 2. $\Gamma = \Gamma_0 \ltimes \mathbb{Z}$, where $\mathbb{Z}$ acts through the shift. Therefore Γ is an extension of an a-T-menable group with amenable quotient, so Γ is a-T-menable by Theorem 2.2 . $\square$

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